JENS HØYRUP

ANOTHER CASE OF STUMBLING PROGRESS IN THE HISTORY OF ALGEBRA

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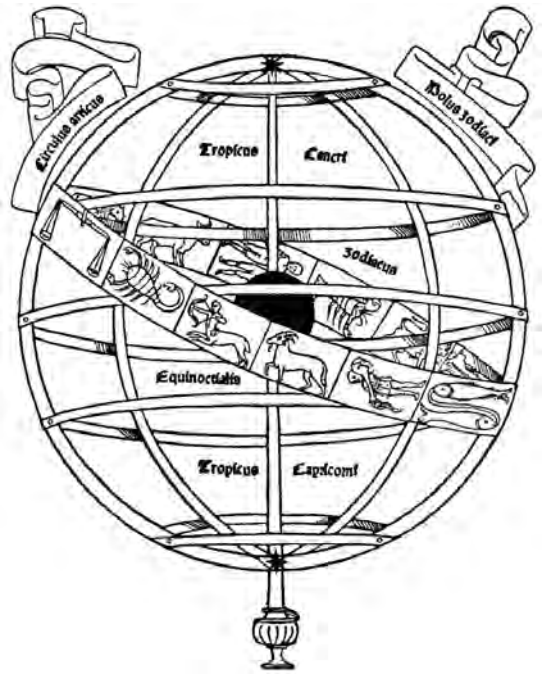

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STUDI E RICERCHE

EMBEDDING: ANOTHER CASE OF STUMBLING PROGRESS IN THE HISTORY OF ALGEBRA

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ABSTRACT – On an earlier occasion I have argued that the development toward full algebraic symbolism in Europe was a case of "stumbling progress," never really intentional before Viète. Here I shall concentrate on a particular aspect of algebraic symbolism, the one that allowed Cartesian algebraic symbolism to become the starting point not only for theoretical algebra, but for the whole transformation of mathematics from his time onward: the possibility of embedding, that is, of making a symbol or an element of a calculation stand not only for a single number, determined or undetermined, but for a whole expression (which then appears as an algebraic parenthesis).

From the Italian beginnings in the fourteenth century, and also in Ibn al-Yāsamin's first creation of the Maghreb letter symbolism, the possibility of embedding was understood and explained in the simple case where a fraction line offered itself as defining a parenthesis; Diophantos, without a line, did something similar on at least one occasion. However, only Chuquet and Bombelli would explore some of the possibilities beyond that, and Viète still less. Even Descartes did not take full advantage of it.

A final section argues why this stumbling character of development should not bewilder us, considering the character of the mathematical practice in which medieval and Renaissance algebra participated.

In memory of Ivor Grattan-Guinness,
friend and wise colleague, master of polemics.

APPROACHES TO ALGEBRAIC SYMBOLISM*

As is well known, Georg Nesselmann's *Algebra der Griechen* suggested a classification of algebra types into three groups: rhetorical, syncopated,

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* The first version of the paper was presented at the workshop *Mathematics in the Renaissance: Language, Methods, and Practices*, ETSEIB, Barcelona, 23 January 2015. I thank for the invitation, and also thank Maria Rosa Massa Esteve for commentaries.

and symbolic.[1] In "rhetorical algebra," everything in the calculation is explained in full words. "Syncopated algebra" uses standard abbreviations for certain recurrent concepts and operations, while "its exposition remains essentially rhetorical."[2] In "Symbolic algebra" (as known to us as well as to Nesselmann), "all forms and operations that appear are represented in a fully developed language of signs that is completely independent of the oral exposition." He also characterized these types as "stages" (*Stufen*), a term that normally indicates chronology; but it is clear from his examples that this division into chronological stages is at most meant locally, not as steps of some universal history.

According to Nesselmann, the rhetorical stage is represented by Iamblichos, by "all so far known Arabic and Persian algebraists," and by all Christian-European writers on algebra until Regiomontanus. Diophantos, and the later Europeans until well into the seventeenth century are classified as syncopated, "although already Viète has sown the seeds of modern algebra in his writings, which however only sprouted some time after him."

In the following pages, Nesselmann mentions Oughtred, Descartes, Harriot and Wallis as creators of this modern, symbolic algebra:

> However, we Europeans since the seventeenth century are not the first to have attained this level; indeed, the *Indian* mathematicians anticipate us in this domain by many centuries.

Probably because they are used by most of those historians who refuse to see *every* use of algebraic abbreviations as a "symbolism," Nesselmann's categories have often been criticized – obvious mistakes or platitudes from the 1840s would have been forgotten long ago. What follows may be read as an attempt to elaborate, substantiate and revise what Nesselmann says in a couple of pages.

As if we all knew and agreed upon what symbolic algebra is, Nesselmann's central observation about what characterizes the symbolic level

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[1] NESSELMANN, 1842, p. 302. This was announced as the first volume of his *Versuch einer kritischen Geschichte der Algebra. Nach den Quellen bearbeitet* – but was, as it turned out, the only volume to appear. In 1843 he published an edition and translation of an Arabic practical arithmetic, after which followed work on Baltic languages (an Indo-European group which he named) and a Sanskritist and Arabist chair. As many Orientalists of his day, he was thus versed in all the languages required for the topic as it could be studied at the time – Latin, Greek, Arabic, Sanskrit, as well as modern European languages. We may envy him.

[2] Here and in the followings, all translations into English are mine if nothing else is indicated.

has mostly been neglected: namely that symbolization allows operations directly on the level of the symbols, without any recourse to thought carried by spoken or internalized language – indeed, almost without recourse to reflective thought. In Nesselmann's words:

> We may execute an algebraic calculation from the beginning to the end in a fully intelligible way without using one written word, and at least in simpler calculations we only now and then insert a conjunction between the formulae so as to spare the reader the labour of searching and reading back by indicating the connection between the formula and what precedes and what follows.

That is exactly what we do when we reduce an equation by additions, divisions, differentiations, and whatever else we may need to apply. We can of course speak *about* the operations we perform, just as we may speak about the operations we perform when changing the tyre on a bicycle or preparing a sauce; but in all three cases *the operations themselves are outside language*.

To illustrate this we may look at two instances of incipient symbolic operation – one from Diophantos, the other from the Italian fourteenth century.

In the *Arithmetic*, Diophantos uses abbreviations (spoken of as "signs" [σημεῖον]) for the unknown number (the *arithmós*) and its powers.[3] The unknown itself is written with a simple sign, something like ς; for the higher powers (*dynamis* = ς^2, *kybos* = ς^3, *dynamodynamis* = ς^4, etc.), phonetic complements are added (Δ^Y, K^Y, etc.); similarly, complements are added to the sign for the monad ("power zero"), and for numbers occurring as denominators in fractions,[4] except in the compact writing of fractions where $\frac{5}{16}$ means $\frac{16}{5}$. Addition is implied by juxtaposition, subtraction and subtractivity are denoted by the abbreviation ⋀ (λεῖψις, "missing" etc.). Only one sign occasionally serves direct operation: the designation of the "part denominated by" *n* (better, indeed, since *n* is not always integer, the reciprocal of *n*); the introduction explains it to be indicated by a sign $^\times$ for powers of the unknown. In III.xi[5] we see that a number which was posited to be $\varsigma^\times$ is stated immediately to be $\frac{41}{77}$ ($\frac{77}{41}$) when ς itself turns out

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[3] Manuscripts do not agree about when and when not to use an abbreviation, but all use them; Diophantos' introduction leaves no doubt that they are really his, and no later scribal invention.

[4] In I.23 (ed. Tannery, 1893, I, p. 92, appears as $\bar{\nu}$ $\kappa\gamma^{ov}$, "50 of 23rds," and slightly later as "150 of the said part").

[5] Ed. Tannery, 1893, I, p. 164.

to be $\frac{77}{41}$. This would hardly have been possible if Diophantos had not known at the level of symbols (and supposed his reader to recognize) that $(\varsigma^{\times})^{\times} = \varsigma$, and that $\left(\frac{q}{p}\right)^{\times} = \frac{p}{q}$. But this, as far as I have noticed without having worked systematically on the text, is the only instance of genuine symbolic operation.

Let us next look at a Tuscan *Trattato dell'alcibra amuchabile*, a compound in three parts from c. 1365.[6] In the third part we find the request to divide 100 first by a "quantity" and then by the "quantity" plus five.[7] The sum of the quotients is told to be 20. So, you should first divide 100 by a *cosa* ("a thing"), and next by "a *cosa* and 5," and join the two quotients. Similar problems (though with subtraction) are found in al-Khwārizmī's and Abū Kāmil's algebras,[8] and again in Fibonacci's *Liber abbaci*.[9] Al-Khwārizmī gives a purely numerical (but sensible) prescription for the initial, difficult steps – obviously, what he did went beyond his technical vocabulary; Abū Kāmil uses a geometric diagram; and Fibonacci applies proportions. The fourteenth-century treatise, however, comes close to what we would do:

> Now I want to show you something similar so that you may well understand this addition, and I shall say thus: I want to join 24 divided by 4 to 24 divided by 6, and you see that it should make 10. Therefore write 24 divided by 4 as a fractions, from which comes $\frac{24}{4}$. And posit similarly 24 divided by 6 as a fraction. Now multiply in cross, that is, 6 times 24, it makes 144; and now multiply 4 times 24, which is above the 6, it makes 96, join it with 144, it makes 240. Now multiply that which is below the strokes, that is, 4 times 6, it makes 24. Now you should divide 240 by 24, from which 10 should result.

Then follows the application:

> Now let us return to our problem. Let us take 100 divided by a *cosa* and 100 divided by a *cosa* and 5 more, and therefore posit these two divisions as if they were fractions, as you see hereby.

$$\frac{100}{\text{per una cosa}} \qquad \frac{100}{\text{per una cosa e piu 5}}$$

[6] The first part contains the sign rules and teaches operations with roots and binomials; the second gives the rules, mostly provided with examples, for the basic "cases" (equation types) until the fourth degree (some of them false); the third, finally, is a problem collection.

[7] Ed. Simi, 1994, pp. 41*f*.

[8] Ed. Hughes, 1986, p. 255; ed. Rashed, 2013, pp. 352-354.

[9] Ed. Boncompagni, 1857, p. 413.

And now multiply in cross as we did before, that is, 100 times a *cosa*, which makes 100 *cose*. And now multiply along the other diagonal, that is, 100 times a *cosa* and five, it makes 100 *cose* and 500 in number; join with 100 *cose*, you get 200 *cose* and 500 in number. Now multiply that which is below the strokes, one with the other, it makes a *censo*[10] and 5 *cose* more. Now multiply the results, that is, 20 against a *censo* and 5 *cose* more, it makes 20 *censi* and 100 things more, which quantity equals 200 things and 500 in number.

This text, we see, is purely rhetorical – everything is written out in full words. On the other hand, the solution proceeds by means of formal operations, in a way we are accustomed to in symbolic algebra; rhetorically expressed polynomials are dealt with *as if* they were the numbers of normal fraction arithmetic. We may say that the *lexicon* of the text is rhetorical, but its *syntax* (in part) symbolic.[11]

Characteristic for this syntax is the phenomenon of *embedding*: the insertion of something possibly complex in the place of something simpler. We know the phenomenon from ordinary language making use of subordinate clauses: *I go <u>now</u>* → *I go <u>when it pleases me</u>*. In contemporary symbolic mathematics indefinitely nested embedding is possible – for instance, in continued fractions, or in the graphically simpler expression:

$$1 + \frac{x}{2} \cdot\left(1 + \frac{x}{3} \cdot\left(1 + \frac{x}{4} \cdot\left(1 + \frac{x}{5} \cdot\left(1 + \frac{x}{6} \cdot(\ldots)\right)\right)\right)\right).$$

In ordinary language, the same possibility is present, restricted only by pragmatic considerations of comprehensibility – "This is the man all tattered and torn / That kissed the maiden all forlorn / That milked the cow with the crumpled horn / That tossed the dog / That worried the cat / That chased the rat / That ate the cheese / That lay in the house / that Jack built."

We may now turn back to Nesselmann. As we remember, he ascribed to the Indian mathematicians a symbolic algebra that precedes that of Europe by many centuries.

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[10] The *censo* is the square on the *cosa*.

[11] (An aside:) And why not? As pointed out by André Weil in a famous polemical note that deserves to be read for much more than its venomous concluding paragraph, "words, too, are symbols" (Weil, 1978, p. 92). *We,* when reducing "3 *things* and two added equal 17" into "3 *things* equal 15" probably use our training in letter algebra, that is, use the syntax of symbolism, stepping outside the framework of grammatical language and forgetting for a while to think of that which the words stand for. A genuinely rhetorical solution would follow the principles of Euclid's common notions (if only at the intuitive level): "But then, since removing equals from equals gives equals, 3 *things* alone must equal 17 with 2 removed," etc. Whether an algebraic text becomes truly rhetorical or hiddenly symbolic depends in part on the reader.

We may look at an example, borrowed from Bhaskara II (b. 1115) via B. Datta and A.N. Singh.[12] What we would express:

$$5x+8y+7z+90 = 7x+9y+6z+62$$

is written by Bhāskara as a scheme:

yâ 5 *kâ* 8 *nî* 7 *rû* 90
yâ 7 *kâ* 9 *nî* 6 *rû* 62

while our

$$8x^3+4x^2+10y^2x = 4x^3+0x^2+12y^2x$$

appears as

yâ gha 8 *yâ va* 4 *kâ va yâ.bhâ* 10
yâ gha 4 *yâ va* 0 *kâ va yâ.bha* 12

Datta and Singh quote David Eugene Smith for the stance that this notation is "in one respect [...] the best that has ever been suggested," namely because it "shows at a glance the similar terms one above the other, and permits of easy transposition."[13]

However, the Indian schemes do not permit direct multiple embedding – for instance the replacement of *yâ* by a polynomial. Nor are they meant for that, they serve exclusively for reducing one side of an equation to zero. The rest of the argument (the initial part that precedes the scheme as well as that based on the reduced equation) is as syncopated as that of Diophantos, albeit with a more systematic use of the abbreviations (and operating with several unknowns) – see the chapter "Varieties of Quadratics" in Bhāskara's *Vijā-ganita*.[14] Replacing a simple by a composite expression requires the same amount of thinking in the Indian notation as in a rhetorically expressed algebra. It is not impossible in either case.

Indian schemes allow certain direct operations, and in this sense they clearly constitute a symbolism, as claimed by Nesselmann. However, Smith is right that this notation is the best "in one respect" only – namely for linear reductions within the restricted framework of problem types actually dealt with by Bhāskara. It allows operations directly at the level of symbols, but only a rather limited, non-expandable set of operations.

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[12] DATTA, SINGH, 1962, II, pp. 31*f*.
[13] SMITH, 1923, II, pp. 425*f*.
[14] Ed., trans. COLEBROOKE, 1817, pp. 245-267.

Stumbling Progress toward Algebraic Symbolism

On an earlier occasion[15] I have described the slow development of algebraic symbolism, from the first introduction in late twelfth-century Maghreb to the final unfolding around Viète and Descartes – not only "hesitating," as my title said, but stumbling. A summary will be useful for the following.

At some moment mathematicians in the Islamic West (the Maghreb, in the general sense including also al-Andalus) invented not only the writing of fractions with a stroke (taken over in the Latin *Liber mahameleth*, plausibly from the 1160s) but also notations for composite fractions, most important the notation for ascending continued fractions such as $\frac{e\ c\ a}{f\ d\ b}$ meaning $\frac{a}{b}+\frac{c}{bd}+\frac{e}{bdf}$ (they are used in Fibonacci's *Liber abbaci*, almost certainly already in the lost first version from 1202).

Probably towards the very end of the century (Fibonacci seems not to know about it), an algebraic symbolism was created, with symbols for powers zero to three of the unknown, and signs for subtraction, inverse, square root and equality; ibn al-Yāsamin († 1204) may have been the inventor. It was first described by Franz Woepcke in 1854[16] on the basis of its use by al-Qalaṣādī (fifteenth c.), that is, well after Nesselmann's perspicacious reference to "all so far known Arabic and Persian algebraists." Already Woepcke suspected from ibn Khaldūn's report that the notation might go back to the twelfth century, as now confirmed by scattered occurrences in writings of ibn al-Yāsamin;[17] from these early traces it is not clear whether the full system we know from later centuries was there from the beginning. In this full system, signs for the powers are written above their coefficient, the root and inverse signs above their argument. The signs are derived from the initial letters of the corresponding words but provided with tails enabling them to cover composite expressions, that is, to delimit algebraic parentheses; the notation served to write polynomials and equations, and even to operate on the equations.

The phrase "algebraic parentheses" asks for two observations. Firstly, a parenthesis is not a (round, square or curly) bracket nor a pair of brackets but an expression that is marked off, *for example* by a pair of brackets; in

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[15] Høyrup, 2010.

[16] Woepcke, 1854.

[17] See Abdeljaouad, 2002, pp. 20, 24*f*.

spoken language, pauses may mark off a parenthesis in the flow of words, and in written prose these are often rendered as a pair of dashes. An *algebraic parenthesis* is an expression marked off as a single entity that can be submitted as a whole to operations; in calculation it has to be determined first. When division is indicated by a fraction line, this line delimits the numerator as well as the denominator as parentheses if they happen to be composite expressions (for instance, polynomials). Similarly, the modern root sign $\sqrt{}$ marks off the radicand as a parenthesis.

Secondly, it is to be observed that the Maghreb notation, though possessing the parenthesis function, does not exploit it fully. More on this below.

The early evidence is accidental, but later extant Maghreb writings are sometimes systematic in their use of the notation, showing that at least its fully developed form can be regarded as a genuine symbolism at the Indian level (though so different in character that influence one way or the other can be safely disregarded).

In these later writings, the symbolic calculations are as a rule made separately from the running text (as can be seen in Woepcke's translation of al-Qalaṣādī, usually following after a phrase "its image is" and thus illustrating the preceding rhetorical exposition. They can also stand as marginal commentaries, as in the "Jerba manuscript" (written in Istanbul in 1747) of ibn al-Hā ʾim's *Šarḥ al-Urjūzah al-Yasminīya*, "Commentary to al-Yāsamin's *Urjuza*" (originally written in 1387).[18] Such marginal calculations probably correspond to what was written on a *takht* (a dustboard, in particular used for calculation with Hindu numerals) or a *lawha* (a clayboard used for temporary writing).[19]

Fibonacci, as stated, does not know the Maghreb notation (his copious use in non-algebraic contexts of rectangular schemes rendering what would be written on a *lawha* makes it almost certain he would have used it if he had known about it). Nor does the earliest generation of abbacus algebra as represented by Jacopo da Firenze's *Tractatus algorismi*.[20] Even algebraic

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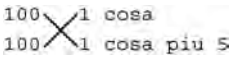
[18] Ed. ABDELJAOUAD, 2004.

[19] See LAMRABET, 1994, p. 203, and ABDELJAOUAD, 2002, pp. 27, 34*f*.

[20] Ed. HØYRUP, 2007a. There are strong reasons to suppose that this algebra, present in only one of the three manuscripts, belongs to Jacopo's original work; but even if it should be a secondary insertion, its closeness to the second section of the *Trattato dell'alcibra amuchabile* (above, note 6) and the way the two texts are reflected in Paolo Gherardi's *Libro de ragioni* from 1328 shows that it must antedate the latter treatise (*ibid.*, pp. 23-25, 163*f*) and hence all other extant vernacular algebra texts.

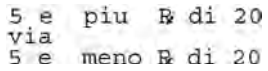
abbreviations are absent in this earliest phase, although abbreviations are of course used profusely in the writing of current words.

Soon, however, some traces of symbolic operation turn up. Paolo Gherardi's *Libro de ragioni* from 1328[21] describes operations on a diagram (itself missing in the copy, which also has a defective text on this point,[22] but which is found in a parallel text[23]):



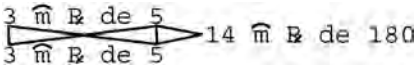
The context is the same problem as discussed above, just after note 6. Clearly, the same operations are thought of, even though the diagram is more rudimentary.

In the first part of the *Trattato dell'alcibra amuchabile*, schemes are used to teach the multiplication of binomials – for example (we now observe the abbreviation ℞ for *radice*, "root"):



The binomials are numerical, but since al-Khwārizmī irrational roots had been used so to speak as pedagogical stand-ins for algebraic roots (square roots of the *censo*, that is, *cose*).

The *Trattato dell'alcibra amuchabile* was written in c. 1365, but even this part of its material is probably older. In Dardi of Pisa's *Aliabraa argibra* from 1344,[24] we find something similar though more elaborate:



Here we find a supplementary abbreviation, m̂ for *meno*, "less." Dardi indeed uses abbreviations systematically: *radice* is always ℞, *meno* ("less") is

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[21] Ed. ARRIGHI, 1987, p. 101.

[22] Unless, of course, Gino Arrighi copies the manuscript badly. However, I doubt that Arrighi would first read *parto* as *porto*, then omit *e poi parto 100 in più 5 che prima* (or something similar), and finally also omit a diagram spoken of in the text.

[23] Florence, Ricc. 2252, see VAN EGMOND, 1978, p. 169.

[24] I use the manuscript Vatican, Chigi M.VIII.170, written in Venetian in c. 1395, checking with van Egmond's personal transcription of a manuscript from 1429 actually held by Arizona State University Temple, for access to which I am grateful. In some of the details, the Arizona manuscript appears to be superior to the others, but at the level of overall structure the Chigi manuscript is demonstrably better (HØYRUP, 2007a, p. 169*f*). Considerations of consistency suggests it to be better also in its use of abbreviations and other quasi-symbolism, for which reason I build my presentation on this manuscript (cross-checking with the transcription of the Arizona-manuscript – differences on this account are minimal); for references I use the original foliation – here fol. 6r.

$\widehat{m}$, *cosa* is *c*, *censo* is *Ç*,[25] *numero/numeri* are *nũo/nũi*. *Cubo* is unabridged, *censo de censo* (the fourth power) appears not as *ÇÇ* but as *Ç de Ç* (an expanded linguistic form which we may take as an indication that Dardi thinks in terms of abbreviation and nothing more). Roots of composite entities are written by a partially rhetorical expression, for instance (fol. 9v) "℞ *de zonto* $\frac{1}{4}$ *cô* ℞ *de* 12" (meaning $\sqrt{\frac{1}{4}+\sqrt{12}}$; *zonto* corresponds to Tuscan *gionto*, "joined;" that a root is "joined" means that it is taken of composite expression, mostly a binomial).

Algebraic monomials are written in a way which we might be tempted to see as an inversion of the Maghreb notation – for instance, "4 *cose*" is written $\frac{4}{c}$. The same notation is used in the original manuscript of the *Trattato di tutta l'arte dell'abbacho* from 1334.[26] Closer inspection of the use reveals, however, that the notation must be understood as a mere reflection of the spoken form, in analogy with the frequent writing of the ordinal *il terzo* as "il $\frac{1}{3}$" (for example number three of "three men") – that is, the fraction notation itself is not understood as an indication of division but as a way to write the ordinal form of the numeral. Even though Dardi was indubitably the best abbacus mathematician of his days and the first to write a treatise dealing solely with algebra, and more consistent in his use of algebraic standard abbreviations than anybody else in his century, he saw no point in exploring the possibilities of symbolic operations.

All in all, until the mid-fourteenth century the only symbolic operations we find are those on formal fractions and the multiplication of binomials in schemes – both rather rudimentary, the former plausibly inspired from Maghreb practices, the latter perhaps an independent development. Algebraic abbreviations remained abbreviations and nothing more, and only Dardi used them systematically.

In the early fifteenth century, the use of standard abbreviations (*co* and *ce*) for *cosa* and *censo* become common (but more often used in marginal annotations than in the running text, rarely very systematically, and very rarely for symbolic operations); they are often written above the coefficient, which might suggest inspiration from Maghreb ways. The first trace of

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[25] Dardi probably thinks of the spelling *çenso*, which corresponds to orthographic habits of his times in north-eastern Italy. In the fifteenth century it was to become *zenso*, which explains the terms and abbreviations of German cossic algebra.

[26] Florence, Biblioteca Nazionale Centrale, fond. princ. II.IX.57. For the dating and for reasons *not* to ascribe the work to Paolo dell'Abbaco, see CASSINET, 2001.

such recent interaction is the algebra section of a *Tratato sopra l'arte della arismetricha* written in Florence around 1390.[27] Probably indirect contact of some kind with the Arabic world is suggested by the use of *censo* for an amount of money which the compiler (in spite of being apparently an extraordinary mathematician) does not understand – after having found the *censo* he takes its square root, believing it has to be an algebraic square, and then has to multiply it by itself in order to find the unknown amount. Beyond sophisticated use of polynomial algebra in the transformation of equation types, we find here a clear discussion of the sequence of algebraic powers as a geometric progression, to which we shall have to return.

The running text contains no abbreviations and certainly nothing foreshadowing symbolic operations. Inserted to the left, however, we find a number of schemes explained by the text and showing multiplication of polynomials with two or three terms (numbers, roots and/or algebraic powers).

Those involving only binomials are related to those of the *Trattato dell'alcibra amuchabile* and Dardi. The schemes for the multiplication of three-term polynomials are of a different kind. They emulate the scheme for multiplying multi-digit numbers, and the text itself justly refers to multiplication *a chasella* as the model.[28] The *a casella* algorithm (roughly identical with ours) differs only from the older *a scacchiera* algorithm, used in the Maghreb multiplication of polynomials (see the "Jerba manuscript"),[29] by using vertical instead of slanting columns.

Such schemes (and other schemes for calculation with polynomials) turn up not only in later abbacus writings (for instance, in Raffaello Canacci's *Ragionamenti d'algebra*,[30] on which more below) but also in numerous sixteenth-century algebras – for example, Stifel's *Arithmetica integra*,[31] Jacques Peletier's *L'Algèbre*[32] and Petrus Ramus's *Algebra*.[33] On the other hand, schemes *of this type* are absent from the three major "abbacus encyclopediae" from c. 1460, all three Florentine and in the tradition reaching back via Antonio de' Mazzinghi (c. 1353 to c. 1391[34]) to Paolo

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[27] Biblioteca Nazionale Centrale, fondo princ. II.V.152. Its algebraic section was edited in FRANCI, PANCANTI, 1988.

[28] Ed. FRANCI, PANCANTI, 1988, p. 9.

[29] Ed. ABDELJAOUAD, 2002, p. 47.

[30] Ed. PROCISSI, 1954, p. 319 and *passim*.

[31] STIFEL, 1544, fols. 3^{v}*ff*.

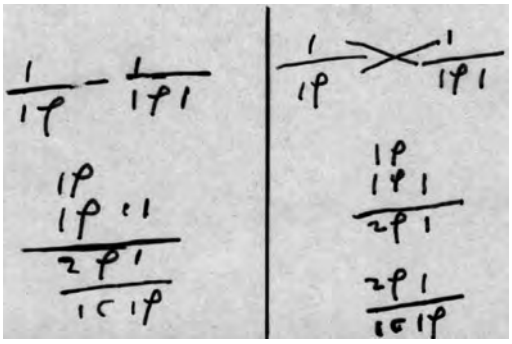
[32] PELETIER, 1554, pp. 15-22.

[33] RAMUS, 1560, fol. A iii^{r}.

[34] ULIVI, 1996, pp. 110*f*.

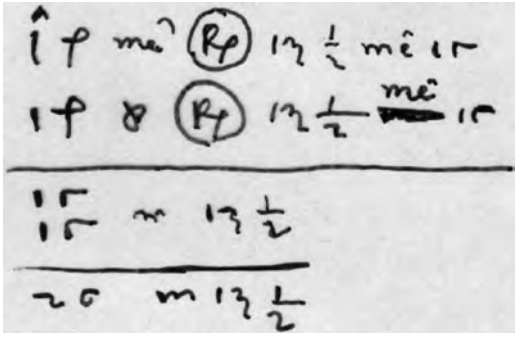
dell'Abbaco and Biagio "il vecchio" (respectively mid- and early-mid-fourteenth c.). Most famous and known from many copies is Benedetto da Firenze's *Trattato de praticha d'arismetrica*. The other two (both known only from the autograph) are Florence, Palatino 573, and Vatican, Ottobon. lat. 3307 – the compilers of the latter two being both pupils of a certain Domenico d'Agostino *vaiaio*.

On the other hand, here we find marginal schemes of this type:



A marginal calculation accompanying the same problem from Antonio's *Fioretti* in Siena, Biblioteca Comunale degli Intronati, L.IV.21, fol. 456^{r} and Ottobon. lat. 3307, fol. 338^{v} (both redrawn).

The appearance of the scheme in similar shape in the different encyclopediae suggests that it goes back to Antonio (from whom the problem itself is borrowed). We also find an abundance of formal fractions, and schemes of a different kind for the multiplication of binomials (ρ stands for *cosa*, *c* for *censo*):



Benedetto's multiplication of (1ρ–√[131 –1c]) by (1ρ+√[13½ –1c]). Redrawn after the autograph Siena, Biblioteca Comunale degli Intronati, L.IV.21, fol. 455^{r}.

All in all, as I once summarized the matter:[35]

> The three encyclopediae confirm that no systematic effort to develop notations or to extend the range of symbolic calculation characterizes the mid-century Italian abbacus environment – not even among those masters who, like Benedetto and the compiler of Palat. 573, reveal scholarly and Humanist ambitions [...]. The experiments and innovations of the fourteenth century – mostly, so it seems, vague reflections of Maghreb practices – had not been developed further. In that respect, their attitude is not too far from that of mid-fifteenth-century mainstream Humanism.

As Humanism, the character and use of notations underwent some changes toward the end of the century – and not only as a consequence of printing (the notational innovations are also found in manuscripts, and sometimes they are more thorough there).

Firstly, the use of abbreviations becomes more systematic, and there is some exploration of alternative systems; secondly, the character of the sequence of powers as a geometric series is taken note of more often, and the sequence of powers is linked to the natural numbers. Sometimes the numbering coincides with our exponents, but the most influential work – Luca Pacioli's *Summa* – makes the unfortunate choice to count *number* as level 1, and *cosa* as level 2 (etc.). In consequence, it still asks for thinking to see that an equation involving (for example) *censi di censo*, *censi* and *numero* is simply a quadratic equation with unknown *censo*.[36]

We still find schemes for multiplication of binomials, sometimes like those of Dardi, sometimes similar to Benedetto's, and also symbolic marginal calculations similar to what Benedetto and his contemporaries had offered – but hardly anything that goes beyond them.

We may jump – in time, and also socially, namely to a scholar treating in Latin of abbacus mathematics *von höheren Standpunkt aus* – to Cardano's *Practica arithmetice, et mensurandi singularis*.[37] Here, we find not only indented marginal schemes (in Benedetto style) but also compact writings in the running texts – a very simple case is the statement (C vii^{r}) that "ducendo ℞.8 ad ℞℞. fit ℞℞.64" ("reducing root of 8 to root of

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[35] Høyrup, 2010, p. 39.

[36] In contrast, the manuscript Modena, Biblioteca Estense, ital. 578 (a copy from c. 1485 of an earlier but probably not much earlier original), whose numerical *gradi* coincides with exponents, classifies higher-degree equations according to the quadratic equations to which they correspond, and apparently feels no need to waste words on the matter.

[37] Cardano, 1539.

root makes root of root of 64"); somewhat more complex (D i^{r}) "1.co.p̃. $\frac{1.men.\ 1.co.}{1.ce.piu.1.}$," meaning *1 cosa* + $\frac{1\ -\ 1\,cosa}{1\,censo\ \mid\ 1}$. "Plus," we observe, may appear both as p̃. and as *piu*. As we shall see below, the use of the parenthesis function is even less systematic in Cardano's *Ars magna* from 1545. It is doubtful whether this can have assisted symbolic operations, and even whether it has supported thought better than full writing (as the marginal schemes indubitably do, but only for the addition, subtraction and multiplication of binomials, which they had always served).

Tartaglia's *Sesta parte del general trattato*[38] is not very different in its use of notations: there are schemes for the operations on binomials, still in Benedetto's style (trinomials are treated stepwise, the *a casella* scheme for polynomials from the *Tratato sopra l'arte della arismetricha* seems to have been forgotten). We also find formal fractions like $\frac{240\,ce.\ men\ 48000}{1\,ce.\ p\ 14\ co.\ p\ 60}$ (fol. 23^{r})[39] and other expressions using abbreviation used in the running text – but nothing with suggests thought supported by symbolic operations.

Michael Stifel, in the *Arithmetica integra*,[40] as already Christoph Rudolff in the *Coss*,[41] use the modern symbols +, – and √, but without making any other changes.

Noteworthy innovations are to be found in the works of Chuquet and Bombelli, but since these innovations are central to our topic we shall deal with them below.

Powers

Let us now return, not so much to embedding as a mere fact as to the willingness to *think in terms* of embedding. This willingness is revealed by the ways in which higher powers were named.

Diophantos introduces these terms for the powers of the unknown:[42]

ἀριθμός (first power)
δύναμις (second power)

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[38] Tartaglia, 1560.

[39] Actually, the *p* standing for *piu* is encircled.

[40] Stifel, 1544.

[41] Rudolff, 1525.

[42] Ed. Tannery, 1893, I, pp. 2-6. Hippolytos refers to the same sequence and names in his *Refutation of all Heresies*, I.2.10 and IV.51.8 (ed. Wendland, 1916, pp. 6, 75). Since Diophantos speaks of the terms as "having been approved" (ἐδοκιμάσθη), this is hardly astonishing.

κύβος (third power)
δυναμοδύναμις (fourth power)
δυναμόκυβος (fifth power)
κυβόκυβος (sixth power)

Obviously, juxtaposition here means multiplication. Nothing in the grammatical construction would suggest otherwise, the nouns are glued together in the standard way to make compositions.

Arabic algebra is very similar. A systematic exposition was given by al-Karajī in the *Fakhrī*:[43]

jidhr or *šai'* (first power)
māl (second power)
ka'b (third power)
māl māl (fourth power)
māl ka'b (fifth power)
ka'b ka'b (sixth power)
māl māl ka'b (seventh power)
ka'b ka'b (eighth power)
ka'b ka'b ka'b (ninth power)

"and so on, until infinity;" indeed, the system allows naming of all powers. Juxtaposition once more stands for multiplication. Grammatically, the connection between the nouns is a genitive, but the Semitic genitive does not, like its Indo-European namesake, necessarily imply a subordination. As we shall see, the use of the Latin and Italian genitive was in the long run to enforce a reading of "the cube *of* the cube" as the ninth power – that is, in modern terms, an understanding of the cube as *a function*, not as an entity.

Only in the long run, however. The Latin translations of al-Khwārizmī's algebra have no names for powers beyond the second (various biquadratics and other easily reducible higher-degree equations are reduced without names being given to the higher powers, and such names therefore do not turn up in the problems). The *Liber mahameleth* refers twice to the *cubus*, explains the first time that the *cubus* is the product of the *census* and its root,[44] but goes no further in the sequence.

Fibonacci, however, does. He does not explain the names nor *a fortiori* the whole sequence, but in the *Liber abbaci* he makes use of those which

[43] Woepcke, 1853, p. 48.
[44] Ed. Vlasschaert, 2010, pp. 338, 363.

he needs.[45] Here we see that the sixth power may be *cubus cubi* as well as *census census census* (this equivalence *is* stated on p. 447, and the latter expression seems to be his standard; we may guess that he follows an Arabic model), while the eighth power is *census census census census*. As we see, he uses the Latin genitive in the Arabic way.[46]

The early abbacus algebras – for instance, Jacopo – go no further than the fourth power, which is *censo di censo* (while the third power is *cubo*). Since 2+2 = 2×2, they can tell us nothing about conceptualizations.

The earliest abbacus writer known to go beyond this boundary is a certain Giovanni di Davizzo. A manuscript written in 1424 (Vatican, Vat. lat. 10488) contains seven pages claimed to be copied from a treatise written in 1339 by him; since the style (use of abbreviations, etc.) is wholly different from what comes before or after, we can probably trust the faithfulness of the copying. The interesting part for our present discussion[47] first gives rules for the multiplication of powers, some of which show the thinking to be multiplicative in spite what might be suggested by the grammar:

> and thing times censo makes cube
> and cube times cube makes cube of cube
> and censo times cube makes censo of cube.

Then follows something which will wring the bowels of any modern mathematician – a daring but mistaken attempt to express negative powers, namely confounding them with roots (the first negative power stated to be "number"):[48]

> And know that dividing number by thing gives number
> and dividing number by censo gives root
> and dividing thing by censo gives number
> and dividing number by cube gives cube root
> and dividing thing by cube gives root
> and dividing censo by cube gives number
> and dividing number by censo of censo gives root of root

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[45] Ed. BONCOMPAGNI, 1857, pp. 447*f*, 450*f*, and *passim*.

[46] In the *Pratica geometriae* (ed. BONCOMPAGNI, 1862, p. 207) *census census* and *cubus cubi* are used in the same way, pp. 214-216 *census census census* and *census census census census*.

[47] Ed. HØYRUP, 2007b, pp. 479-481.

[48] Along with a number of false solutions to cubics and quartics, this system survived until Bento Fernandes' *Tratado da arte de arismetica* from 1555, see DO CÉU, 2008. Maria do Céu's attempt (p. 9) to save the system mathematically is ingenious but disagrees completely with the words and the structure of the various texts that state these rules (and obviously never use them).

and dividing thing by censo of censo gives cube root
and dividing censo by censo of censo gives root
and dividing cube by censo of censo gives number
and dividing number by cube of cube gives cube root of cube root
and dividing thing by cube of cube gives root of cube root
and dividing censo by cube of cube gives root of root
and dividing cube by cube of cube gives cube root
and dividing censo of censo by cube of cube gives root
[…]

It may not be warranted to look for anything in the text beyond a play with words, but we can still try to take it as seriously meant, and suppose that Giovanni's "roots" in this context are intended to be the same as those he speaks about in the first part of the excerpt (which are those of everybody else). Under these conditions we see that even his roots are supposed to be composed "multiplicatively" (whatever can have meant by that) – for instance, that the cube root of the cube root is the sixth, not the ninth root. Similarly, the *Trattato dell'alcibra amuchabile*[49] takes *radicie de radicie chubica* to be the fifth – but since this is once again in a messy context where no calculation is performed, the compiler has no reason to discover that his rule is absurd.

Dardi knows better. His names for the *powers* are still in the Arabic style, and he even explains like Fibonacci that *Ç di Ç di Ç* is the same as *cubi di cubi* (fol. 43[r]). Since most of his problems involve radicals (in the style of "roots of cubes"), he gives us the occasion to observe that he understands roots as functions, and that repeated root taking thus involves embedding – expressing for example (fol. 95[r]) the twelfth root as

$$(\sqrt[3]{\sqrt{\sqrt{a}}} \text{ or } \sqrt{\sqrt[3]{\sqrt{a}}}),$$

while his term for the twelfth power would be *cubo di cubo di cubo di cubo*. But this terminological insight and innovation has a price: Dardi has no name for the fifth and the seventh root, and once replaces the former by ℞ *cuba* (fol. 97[v]), and once the latter by ℞ *dela* ℞ (fol. 98[r]), thus ending up with mistaken rules.[50] In spite of his manifest command of the sequence of powers, he is at the limit of what he can express.

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[49] Ed. Simi, 1994, p. 48.
[50] Cf. van Egmond, 1983, p. 417.

Toward the end of the fourteenth century, the frontier had moved, and the consequences of the genitive construction made themselves felt – but as yet inconsistently.

The manuscript Palat. 573 (one of the three "abbacus encyclopedia" mentioned above) quotes Antonio de' Mazzinghi for the following:[51]

> *Cosa* is here a hidden quantity; *censo* is the square of the said *cosa*; *cubo* is the multiplication of the *cosa* in the *censo*; *censo di censo* is the square of the *censo* [*quadrato del censo*], or the multiplication of the *cosa* in the *cubo.* And observe that the terms of algebra are all in continued proportion; such as: *cosa, censo, cubo, censo di censo, cubo relato, cubo di cubo*, etc.

As we see, Antonio avoids speaking of the fifth power as *cubo di censo* or *censo di cubo*, introducing instead a neologism; but his naming of the sixth power is still multiplicative, and when he suggests an understanding of the fourth power through embedding the name for the function is *square*, not *censo*. The name for the fifth power may have been inspired by his term for the fifth root, appearing in a problem about composite interest[52] as *radice relata*.

The extensive algebra section of the *Tratato sopra l'arte della arismetricha* (Florence, c. 1390, see above) – also from the hand of a highly competent algebraist – starts by explaining how the powers are produced one from the other, and that they are in continued proportion.[53] One particularity is an extra identification of these as "roots," namely (as explained) as the roots which they have.[54] Taking this into account, the sequence is:

cosa (first power)
censo or *radice* (second power)
cubo or *radice cubica* (third power)
censo di censo or *radice della radice* (fourth power)
cubo di censi or *una radice che nascerà d'una quantità quadrata chontro a una quantità chubicata* or (some say) *radice relata* (fifth power)
censo di cubo (sixth power).

For the sixth power it is stated (but not properly given as a name) that one may take the root, and of this quantity take the cube root. The author

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[51] Ed. Arrighi, 2004, p. 191.
[52] Ed. Arrighi, 1967, p. 38.
[53] Ed. Franci, Pancanti, 1988, pp. 3-5.
[54] "Tanto vol dire uno censo quanto dire una quantità ch'à radice" (p. 3); "questa quantità di nome che produce radice relata" (p. 5).

thus recognizes the embedding of the taking of roots, and transfers this to the name *censo di cubo*, corresponding to our $(x^3)^2$; this, however, does not force him to give up the multiplicative name for the fifth power, also identified as "a root born from a squared root multiplied against a cubicated quantity." The name *radice relata* ascribed to "some," we observe, coincides with the name for the fifth root used by Antonio.

Benedetto as well as the compiler of Palat. 573, both of whom copy long extracts from Antonio, also take over his naming in their independent chapters. The third encyclopaedia instead (Ottobon. lat. 3307) uses both *cubo di censo* and *censo di cubo* about the fifth power; the intervening 70 years have thus not witnessed any steps beyond the inconsistencies of the late fourteenth century.

There is some – though not yet really exhaustive – change toward the end of the fifteenth century. Above, the algebra of the Modena manuscript Bibl. Estense, ital. 578 was mentioned for its use of *gradi* coinciding with our exponents. It also uses the "root names" for the powers, and the abbreviations *C*, *Z* and *Q* for *cosa*, *censo* (thought of in the North Italian orthography *zenso*) and *cubo* – in the running text (and once in the scheme below), however, *censo* is represented by a variant of Dardi's *Ç*.[55] The whole sequence (fol. 5^r) is then abbreviated:

N (power zero)
C (first power)
Z (second power)
Q (third power)
ZÇ (fourth power)
CđZZ (fifth power)
ZđQ (sixth power)
CđZđQ (seventh power)
ZđZZ (eighth power)
QđQ (ninth power).

Since, as always, 2+2 = 2×2, we cannot decide the principle according to which the name for fourth power is formed; the fifth, however, is clearly formed from the fourth as a multiplication, whereas the 6th is based on

[55] Namely, with a much enlarged cedilla – of interest only because it is also used by Jacques Peletier (PELETIER, 1554), which shows Peletier to know not only Stifel, Pacioli and Cardano but also at least part of the manuscript tradition.

embedding. The seventh is based on mixed principles, the eighth and the ninth on pure embedding.

On fol. 5^{v}, a new scheme gives the corresponding root significations:

C: *egli che trovi* ("that which you find").
Z: *la R. di quello* ("the root of that").
Q: *la R. quba di quello* ("the cube root of that").
ZÇ: *la R. di R. di quello* ("the root of the root of that").
CD̄ZZ: *la sua R. di quello* ("its root of that").
ZD̄Q: *la sua R. de la R. di quello* ("its root of the root of that").
CD̄ZD̄Q: *la 7^{a} R. di quello* ("the 7th root of that").
ZD̄ZZ: *la R. di R. di R. di quello* ("the root of the root of the root of that").
QD̄Q: *la RQ di la RQ di quello* ("the cube root of the cube root of that").

As we see, there is a strong coupling between the roots that are expressed via embedding and the corresponding powers. The seventh, irreducible root is referred to with this name, whereas the fifth root is unspecified.[56] All in all, a preliminary conclusion suggests itself: Namely that the much more obvious embedding of roots is what started enforcing also the view of power-taking as an embedding (that is, as a function or an operation).

Raffaello Canacci's *Ragionamenti d'algebra* from c. 1495 has idiosyncratic names for some of the higher powers:

numero (power zero)
chosa (first power)
censo (second power)
cubo (third power)
censo di censo (fourth power)
chubo di censo (fifth power)
relato (sixth power)
promico (seventh power)
censo di censo di censo (eighth power)
chubi di chubi (ninth power)
relato di censo (tenth power).

[56] It is possible that the copyist has misread "5^{a}" as "sua."

The fifth power is thus named according to the multiplicative principle, but the eighth and ninth by embedding (the Modena manuscript does the same, but not in the same way for the fifth power). The name for the sixth power is the one others use for the fifth power, and that for the seventh is even more astonishing, and in absolute conflict with normal usage.[57] The name for the tenth power falls outside both systems (but see imminently).

Canacci also experiments with graphic notations for the powers – *censo* is a square, *cubo* a vertically divided rectangle, *censo di censo* two separate squares, his *relato* a horizontally divided rectangle, his *promico* a horizontally divided square. Their compositions emulate those of the names.

According to Francesco Ghaligai,[58] almost the same names and graphic signs had been used by Giovanni del Sodo (Canacci's teacher) in his algebra, with the extension that the 11th power was *tromico*, and the 13th was *dromico*. But del Sodo, according to Ghaligai, used *relato* about the fifth power, and named the sixth power with embedding, as *cubo di censo*. In this system, *relato di censo*, understood as embedding, is really the tenth power. Del Sodo's system is thus consistently based on embedding, although his graphic notation must be characterized as unhandy. Canacci's inconsistencies, it turns out, must be traced back to deficient understanding of his model (his rules for multiplication of powers[59] confirms this). However, such misunderstandings on the part of an otherwise competent abbacus writer shows that del Sodo's way to think was not yet commonplace, nor central to mathematical practice – concepts that are appropriated through use are not mixed up like this by trained practitioners.

In Pacioli's Perugia manuscript from 1478,[60] those 25 sheets are missing where a systematic presentation of the powers would be expected (missing also according to Pacioli's own table of contents). Since the problems do not deal with powers beyond the fourth, we can only see that *cosa*, *censo*, *cubo* and *censo di censo* are represented by superscript ^{co}, $^{\square}$, $^{\Delta}$ and $^{\square\square}$, from which we can conclude nothing.

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[57] *Pronic numbers* are numbers of the form $n\cdot(n+1)$, and the pronic root is related in other authors to this concept, though not always in the same way. According to Pacioli (PACIOLI, 1494, I, fol. 115^{v}), the pronic root of 84 is 9, because $(9^2+\sqrt{9}) = 84$, while Gilio (ed. FRANCI, 1983, pp. 18*f*) as well as Muscharello (ed. CHIARINI ET AL., 1972, p. 163) state that it is 3. Benedetto (ed. PIERACCINI, 1983, p. 26) suggests without being quite clear that the pronic root of 18 is 4, which would agree with Pacioli ($4^2+\sqrt{4} = 18$).

[58] GHALIGAI, 1521, fol. 71^{r}.

[59] Ed. PROCISSI, 1954, p. 433.

[60] Ed. CALZONI, GAVAZZONI, 1996.

We may look instead at his *Summa*,[61] which uses a different notation (plausibly because superscripts were not possible or acceptable for his printer, who seems to have counterbalanced Pacioli's loquacity by reducing line distances to an absolute minimum). Here, fol. 67^{v}[62] lists the 30 *gradi* of the "algebraic characters" or *dignità* (as he says they are called):

℞ *1^{a} n^{o}. numero* (power zero)
℞ *2^{a} co. cosa* (first power)
℞ *3^{a} ce. censo* (second power)
℞ *4^{a} cu. cubo* (third power)
℞ *5^{a} ce.ce. censo de censo* (fourth power)
℞ *6^{a} p^{o}.r^{o} primo relato* (fifth power)
℞ *7^{a} ce.cu. censo de cubo e anche cube de censo* (sixth power)
℞ *8^{a} 2^{o}.r^{o}. secundo relato* (seventh power)
℞ *9^{a} ce.ce.ce. censo de censo de censo* (eighth power)
[...]
℞ *29^{a} ce.ce.2^{o}.r^{o}. censo de censo de secundo relato* (twenty-eighth power)
℞ *30^{a} [9^{o}]r^{o}. nono relato* (twenty-ninth power).

Everywhere, composition means embedding, and the prime powers are designated as 1st, 2nd, 3rd, 4th, ... 9th *relato*.

So, with del Sodo and Pacioli, embedding-composition has become the sole principle. Taking a power, in other words, has become *an operation*, and the power itself more or less a function. And in the Modena manuscript as well as Pacioli, the members of the sequence are identified arithmetically.

Chuquet's *Triparty des nombres* from 1484[63] was more radical. Dropping all names, Chuquet simply wrote the exponent of the power superscript after the coefficient. This was *too* radical, at least in the opinion of Étienne de la Roche, whose *Larismethique nouvellement composee*,[64] based to a large extent on Chuquet's work and the only channel through which Chuquet's ideas reached the wider world, turns instead to the notations that were becoming current in German algebra at the time, for instance in Rudolff's *Coss* (ultimately going back to the Florentine notations of the mid-fifteenth century) – see Barbara Moss's analysis,[65] in particular the

[61] Pacioli, 1494.
[62] Repeated on fol. 143r within a more complicated structure.
[63] Ed. Marre, 1880.
[64] De la Roche, 1520.
[65] Moss, 1988.

comparison between Chuquet's manuscript and de la Roche's corresponding text on p. 122.

Rudolff[66] offers this sequence on fol. D ii^{v} (I omit his graphic symbols):

dragma oder numerus (power zero)
radix (first power)
zensus (second power)
cubus (third power)
zensdezens (fourth power)
sursolidum (fifth power)
zensicubus (sixth power)
bissursolidum (seventh power)
zenszensdezens (eighth power)
cubus de cubo (ninth power)

– also based on embedding, but with new terms for the prime powers, obviously invented in a Latinizing environment (*sursolidum*/*supersolidum* might be related to Antonio's *cubo relato* – a cube, after all, *is* a solid).

The same powers and graphic symbols are given by Stifel in the *Arithmetica integra*[67] – but Stifel goes on until the 16th power (after *zensocubicus* only with graphic symbols). Similarly, Tartaglia, in the *Secunda parte del general trattato de numeri e misure* repeats Pacioli's list[68] – and again in the *Sesta parte*,[69] though stopping here at the 14th power because one very rarely needs so high powers (but pointing out in both volumes that one may go on *in infinito*).

Bombelli, in the manuscript of his *L'algebra*, uses an arithmeticized notation with indication of the power written above the coefficient – for instance, $\overset{1}{30}$ for "30 *cose*,"[70] which in the printed version would become $30\underset{\sim}{1}$. In the beginning of book I,[71] however, he explains the terms which we know from Pacioli and Tartaglia – though only until *numero quadrocubico, over cubicoquadrato*. As Tartaglia in the *Sesta parte* he obviously sees no purpose in discussing, for the sole reason that they can be given a name, powers that are of no use in his work.

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[66] Rudolff, 1525.
[67] Stifel, 1544, fols. 234^{v}-235^{r}.
[68] Tartaglia, 1556, fol. 73^{r}.
[69] Tartaglia, 1560, fol. 1^{r}.
[70] Bortolotti, 1929, p. 21.
[71] Bombelli, 1572, pp. 1-3.

All in all, the insights in this domain that had been reached by del Sodo and Pacioli in the late fifteenth century were conserved and systematized but not superseded during the following century.[72] But how could they be without an explicit parenthesis function allowing the automatization and apparent trivialization[73] of such insights as $x^{(mn)} = (x^m)^n = (x^n)^m$? Therefore, we shall now turn our attention to the algebraic parenthesis.

The Parenthesis before and until the Brackets

Once upon a time there was a "Babylonian algebra." It was discovered (or invented) around 1930, but over the last three decades I believe I have managed to convinced most of those who work seriously on the topic that the numbers found on the tablets and supposed to reflect algebraic operations correspond instead to the measures of geometric entities manipulated in a cut-and-paste technique (whether this technique can then be characterized as "algebraic" is a matter of taste or definition). For instance, let us look at a literal translation of the very simplest second-degree example – the first problem on the tablet BM 13901, a "theme text" about squares[74] (see the box on the next page).

A "confrontation" is the side of a square (which "confronts" its equal), the "moiety" is a "natural half," that is, a half whose role could not be filled by any other fraction. To "make *a* and *b* hold" stands for the construction of a rectangle with sides *a* and *b*, and that *s* "is equal by" *A* means that *s* is the side of the area *A* laid out as a square. The "projection" gives the clue to the method: at first the side or "confrontation" *c* is provided with a "projection," a breadth 1, which transforms it into a rectangle with area

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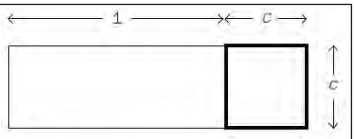
[72] Nor had they totally superseded the multiplicative understanding. Viète's *In artem analyticen isagoge* (Viète, 1591, fol. 4^v) still speaks of the fifth power as *quadrato-cubus* and of the sixth as *cubo-cubus*. The way the compositions are formed, with *-o-*, suggests that this was a conscious return to Diophantine manners, away from the "filthy jargon" of current algebra (*à barbaris defoedata et conspurcata*, as formulated by Viète in the dedicatory letter. p. A ii^v).

Oughtred, 1648, fol. 35*f*, takes over this multiplicative understanding, now also in symbols (*quadrato-cubum* becoming qc). Since Oughtred uses juxtaposition in the modern way, as multiplication, this in also unobjectionable. Still, none of the two has the idea that powers could be functions.

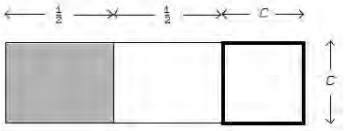
[73] Apparent! Cf. Weil, 1978, p. 92, where exactly this is discussed.

[74] Borrowed from Høyrup, forthcoming. Since the Babylonian sexagesimal place value system is immaterial for the present discussion, I translate the numbers.

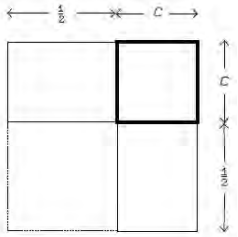
1) The surface and my confrontation I have heaped: $\frac{3}{4}$ is it. 1, the projection,



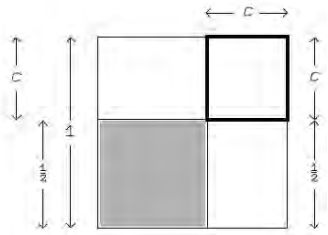
2) you posit. The moiety of 1 you break, $\frac{1}{2}$ and $\frac{1}{2}$ you make hold.



3) $\frac{1}{4}$ to $\frac{3}{4}$ you join: by 1, 1 is equal. $\frac{1}{2}$ which you have made hold



4) from the inside of 1 you tear out: $\frac{1}{2}$ the confrontation.



$1 \cdot c = c$. Then, according to the statement, this rectangle, together with the square $\square(c)$, has a total area $\frac{3}{4}$. Breaking it into two equal parts and moving one of them around we get a gnomon, still with area $\frac{3}{4}$, which is completed by a square of area $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. The completed square has an area $\frac{3}{4} + \frac{1}{4} = 1$ and therefore a side $\sqrt{1} = 1$. Removal of the part which was added below leaves us with the original side, which must hence be $1 - \frac{1}{2} = \frac{1}{2}$.

This technique seems to leave no space for anything like a parenthesis, and at this level the immediate impression holds true. However, the technique may be used for "representation," that is, the sides of square and rectangular areas may themselves be areas, volumes, numbers of working days or bricks produced during these days, prices, etc. In the particular case where the sides of a rectangle are two square areas, we may describe

the solution as making use of an *implicit* parenthesis – as when Fibonacci solves a bi-biquadratic problem in the *Liber abbaci*[75] by treating the *census census census census* as a square area and the *census census* as its side.[76]

The Babylonian texts also present us with an *explicit* parenthesis function, though (in the likeness of the parenthesis demarcated by a fraction line, see imminently) only used in very specific contexts. They make use of two different subtractive operations, *removal* and *comparison*.[77] For an entity *b* to be removed from another entity *B* (for instance, but there are synonyms and almost-synonyms, by being "torn out"), *b* has to be a part of *B*. An entity *A* that is no part of *B* obviously cannot be removed from it, but instead the text may state by how much *B* exceeds *A*. In the former case, the operation produces an entity that can be subjected to the usual geometric operations. In the latter, however, only few texts see the excess as an independent quantity that can be directly manipulated, for instance by constructing a square with the excess as side (making the excess "confront itself"). The majority would make "*so much as* that by which *B* exceeds *A*" confront itself. The phrase "so much as" (translating *mala*, a single word) thus defines a parenthesis.[78] However, the use of this parenthesis is not general, and like implicit parentheses (Babylonian, or Fibonacci's) it cannot be nested without strain on thought. Within the kind of mathematics that was practised (by the Babylonian calculators, or by Fibonacci) it is also dubious whether any use for such nesting would easily present itself.[79]

Because of the tails with which the superscript symbols for powers and root were provided in the Maghreb notation, these symbols may serve to delimit parentheses.[80] The argument of a root sign may be a complex

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[75] Ed. BONCOMPAGNI, 1857, p. 447.

[76] However, Fibonacci takes care here not to identify this square with another *census*, but uses *Elements* II.6 – he is not quite as close to an implicit parenthesis as is the Babylonian text. On p. 422, it is true, a *census* is re-baptized *res* – but it is not clear whether that *census* is meant as an algebraic second power or just renders the original Arabic meaning of *māl*, an amount of money.

[77] This is actually a simplification – HØYRUP, 1993, provides some shades; but it is a close approximation, and sufficient in the present context.

[78] It is also used in slightly different ways – for example, "So much as I have made confront itself, and 1 cubit exceeding, that is the depth" (namely of an excavation with square base).

[79] One example comes to my mind – problem #3 of the text TMS IX (see HØYRUP, 2002, pp. 91-95). Here, the entities at the first level are given new names (the original sides of a rectangle augmented by 1); at the second level, no such names are introduced, the unknown at this level are simply understood to be 3 respectively 21 times those of the first level. All in all, two-level embedding is thus eschewed.

[80] Examples in ABDELJAOUAD, 2002, pp. 23-46.

expression, and may even itself contain roots (nesting). Inverse taking may have an algebraic monomial as its argument, and it may even be repeated; but the notation is ambiguous as regards the coefficient (will it produce $5 \cdot x^{-1}$ or $(5 \cdot x)^{-1}$?). Division written fraction-wise may contain algebraic polynomials in the numerator as well as the denominator.

The situation concerning the symbols for powers is different. Here, the argument (which is actually the coefficient) may be an integer or a broken number or even an arithmetical composite, but nothing else. Number, *šai'*, *māl* and *ka'b* have individual signs; higher powers are written as composites either horizontally or vertically, but the meaning will always be multiplicative (as in al-Karajī's verbal list of their names) – the sign for *māl* and *ka'b* written together will always stand for *māl ka'b* (the fifth power), never for $(x^3)^2$. In other words: *šai'*, *māl* and *ka'b* are *entities*, not functions or operations, as are the root sign and the division written fraction-wise.

All in all: at least in its mature phase the Maghreb notation comprised a fairly well developed parenthesis function – certainly more fully developed than anything that can be found in Europe before and even including Viète; but like Viète[81] it stopped short of the point where it could be used for free symbolic manipulation.

In Latin (that is, Romance and Germanic) Europe, as we have seen, powers remained entities until the mid-fifteenth century; even for del Sodo, Pacioli etc., who consistently named higher powers by embedding, it was still impossible to use their names as operations on other entities than powers of the unknown.

Formal fractions carrying a binomial in the denominator were in use from the mid-fourtenth century, as we have seen; in the fifteenth century, trinomials also appear occasionally. In the beginning, however, this devel-

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[81] As we have seen, even Viète's powers are entities, not functions allowing nesting; his copious use of proportion technique would also make the use of nested expressions almost as difficult as in the Indian notation. And like Bhaskara II, he steps outside symbolic calculation when needing to operate with complex expressions, as for instance in *Ad logisticen speciosam notae priores*, prop. 41 (ed. VAN SCHOOTEN, 1646, p. 32):

> Sit radix binomia A+B, sublaterale coëfficiens D planum. Effingendum sit solidum ub A+B, &D plano, adfectum multa cubi ex A+B. Ducatur A+B in D planum multatum A+B cubo. Orientur solida, A in D planum, +B in D planum, – A cubo, – A quadrato in B 3, – A in B quadratum 3, – B cubo.

As translated in WITMER, 1983, p. 64:

> Let $A + B$ be a binomial root and D^p the coefficient of its first power. The solid from $A + B$ and D^p, and affected by the subtraction of the cube of $A + B$, is to be constructed. Multiply $A + B$ by D^p minus the cube of $A + B$. There then arise these solids: $AD^p + BD^p - A^3 - 3A^2B - 3AB^2 - B^3$.

opment was stymied by the predominant understanding of the fraction line as an indication of ordinality and not of division; no wonder, perhaps, that this borrowing from the Maghreb took a long time to get established.

For roots, the sign R̸ came in use before 1340 (Giovanni di Davizzo used it in 1339, and Dardi in 1344). The cube root, however, was written R̸ *cubo*, and roots of composite expressions also had to be designated "R̸ *de zonto*" (Dardi), ⓇⓍ by Gilio (who may have taken it over from his master Antonio[82]) and also by Benedetto; and R̸ *legata* or R̸u (*u* for *universale* or *unita*) by Pacioli and Cardano. Mostly, but not consistently, this root was to be taken of a binomial; Cardano, moreover, might use R̸u of a binomial as the sum of the two roots (R̸u$(a+b) = \sqrt{a}+\sqrt{b}$).[83] That is, R̸u is no symbol proper but only an abbreviation, whose meaning must be understood from context (as current in manuscript abbreviations, where a stroke over a vowel might mean that either *m* or *n* was to follow, and where the same abbreviation might stand for *phisice* as well as *philosophice*).

So, in spite of the original access to inspiration from the Maghreb and to the enduring use of the algebraic parenthesis defined by the fraction line, the obvious *need* for an unambiguous way to take roots of polynomials, that is, for a delimitation of the radicand as a parenthesis, was only answered by Chuquet, who used the simple trick to underline the radicand.[84] As far as I have noticed, he does not use the notation for other purposes, and it is never nested. De la Roche may have found the innovation superfluous.

For the root of binomials, Bombelli still uses *Radice legata* or *Radice universale*, as he explains.[85] Longer radicands (and sometimes also binomials, for instance on p. 106), are delimited by an initial ⌊. and a final inverted ⌋; sometimes, the system is nested (but always with each parenthesis being a radicand). The lack of system indicates that the purpose is disambiguation and nothing more. Bombelli's manuscript, however, goes somewhat further: the whole radicand is underlined, and the beginning and the end of the line are marked by vertical strokes.[86]

In *La seconda parte del general trattato*, Tartaglia had already used round brackets occasionally to delimit universal roots without explaining – but only when it turns out to be needed for disambiguation, namely

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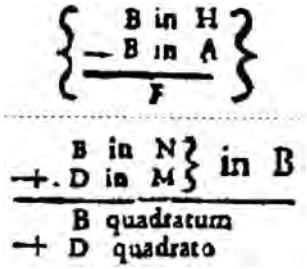
[82] Franci, 1983, p. xxiii.
[83] See the survey of his notations in Tamborini, 2011, p. 57.
[84] Ed. Marre, 1880, p. 734 and *passim*.
[85] Bombelli, 1572, pp. 98*f*.
[86] See Bortolotti, 1929, p. 6.

in the exposition of how to operate with several universal roots, or with universal roots and numbers.[87] Occasionally, only the initial bracket is present. Beyond that, Tartaglia makes generous use of round brackets as punctuation (as already Pacioli had done, yet without getting the idea to borrow them for algebraic purposes).



Viète's suggested parentheses, see note 89.

So, where does the *general* algebraic parenthesis begin? Not yet with Viète. In *Ad logisticen speciosam notae priores*, prop. 53,[88] what we would write $B^2+(Z+D)^2$ is expressed with a rhetorical parenthesis as "B quadrato, + quadrato abs Z+D" – but in the accompanying diagram it appears as "$Z+D^q+B^q$", where ^{q} stands for *quadratum*. This is at least as ambiguous and just as context-dependent as Cardano's ℞u.[89]

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[87] TARTAGLIA, 1556, fol. 167^v.

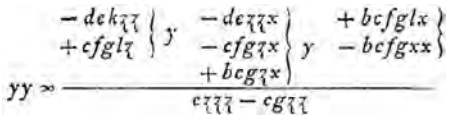
[88] Ed. VAN SCHOOTEN, 1646, p. 38. I have been unable to find the original edition, which however is not likely to have been more consistent.

[89] In *Zeteticorum libri V* (VIÈTE, 1593), a notation is occasionally used which certain seventeenth century readers would understand as an algebraic parenthesis. Inspection of the text shows that this was not Viète's intention. A pair of curly brackets, or a single brace, serve to indicate that an expression going over several lines is meant to kept together typographically rather than mathematically. So, on p. 3^r we find the upper expression in the adjacent diagram (translated $\frac{BH \quad BA}{F}$ in WITMER, 1983, p. 93); VAN SCHOOTEN, 1646, p. 45, sees that there is no need for specification of a parenthesis – the fraction line suffices – and writes $\frac{B \text{ in } H. - B \text{ in } A}{F}$. VASSET, 1630, p. 50, and VAULEZARD, 1630, p. 38, offer something very similar in their translations.

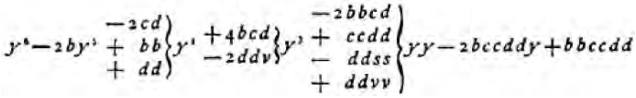
In the original edition of the *Zeteticorum libri V* we see on fol. 18^r that a sole right brace can be used in the same function, and once again van Schooten (p. 70), Vasset (p. 141) and Vaulezard (p. 166) simply write numerator and denominator on a single line each. On Viète's fol. 17^r we see that a single right brace may also stand along the numerator alone – and even here, van Schooten (p. 69), Vasset (p. 138) and Vaulezard (p. 162) simply write the numerator in one line, omitting the brace.

On Viète's fol. 15^r, on the other hand, we find something which the seventeenth-century editors and translators would see differently, even though nothing in Viète's original text suggests he saw any difference – namely the lower expression in the diagram. In this case, van Schooten (p. 65)

What then about Descartes? Firstly, of course, Descartes has the modern long square root $\sqrt{\ }$, which can also be nested – for instance $\sqrt{-\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$.[90] Next, he uses complex expressions involving multiple parentheses, as in this equation (p. 398):



As we see, the parentheses are not enclosed in pairs of brackets, but written vertically and kept together by a brace to the right; but that is immaterial as long as they are unambiguous.[91] We also notice that Descartes prefers to write second powers as *yy*, even though he writes y^3 (etc.), as in this expression (p. 420):



but that, again, is a different question (we too, when dealing with angles, may write 2°23'12"25$^{(3)}$, and similarly differentiate sequentially as f(x), f'(x), f"(x), f$^{(3)}$(x), ...).

Descartes does not use these parentheses very much, but they are there. And as Engels states in *Dialektik der Natur*,[92] "100,000 steam engines [prove the principle] no more than one;" or, at least, in a formulation ascribed to Wilamowitz-Moellendorf, "according to the philologists, once is never, twice is always." Nor did mathematicians of the following generation

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and Vaulezard (p. 139) conserve the brace, seeing that it has a function (the latter with a misprint). Vasset (p. 125) conserves the brace but locates it after the fraction (containing both numerator and denominator), where it is actually superfluous; but he must understood it as having a function in Viète's text.

It is tempting to see this reinterpretation of Viète's text as a reflection of a "cognitive pull" produced by the development of seventeenth-century mathematics: once the need for a more general parenthesis function was there, it made Vasset and Vaulezard read it into the text under their eyes (van Schooten is a different and less significant case: he was close to Descartes [VAN RANDENBORGH, 2012], and he wrote after 1637).

[90] Ed. ADAM, TANNERY, 1897, VI, p. 375.

[91] It is also immaterial that the brace had already been used by Viète in a different function – but perhaps more significant that Vaulezard has changed the use of Viète's notation into something close to what Descartes was doing.

[92] ENGELS, 1962, p. 496.

use Descartes' invention very much.[93] In the *Arithmetica infinitorum*,[94] for instance, Wallis has many complex expressions kept together by fraction lines; but when fractions are written with a slash on a single line and his modern translator[95] writes $(l^5+10l^4+35l^2+50l^2+24l)/5$, there Wallis himself (p. 149) has $l^5+10l^4+35l^2+50l^2+24l/5$. In *Mechanica, sive, De motu*,[96] he even uses an explanatory parenthesis inside a formula – $\frac{6fvR^2-3fs2R(=3fv^2R)-4as^2-3s^3v}{-3eR^2+12avR+9svR-6as^2+4s^3}$ (similarly p. 427). When continuing a numerator or a denominator over two lines (for instance, p. 411) he sees no need to indicate (as had done Viète) that the whole belongs together as one expression. This is not out of ignorance: in his correspondence with John Collins, he uses both the Cartesian brace (namely when presenting something as Descartes' solution to a problem) and round brackets when it fits.[97] But his mathematics asks for no systematic use.

A similar picture is offered by Newton. His Mathematical Notebook from c. 1664/65[98] uses few parentheses beyond the traditional types kept together by fraction lines – but one does find others (thus fol. 152^v), which are delimited by a vinculum (a horizontal line above the parenthesis). This notation is still used in his *Arithmetica Universalis*[99] as well as his writings on fluxions[100] – but sparsely used, and for simple purposes only.[101]

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[93] In Descartes' own generation there are many who see great promises in the expanded use of symbols.

Oughtred's *Clavis mathematica* (OUGHTRED, 1648) was already mentioned. Inasfar as embedding is concerned, Oughtred does not go beyond Dardi or Benedetto (Oughtred's only parentheses are the fraction line, and the "universal root," which he writes √).

Since neither Oughtred nor others go beyond what was done in earlier centuries with parentheses, there is no reason the go into depth with them in the present context, nor to discuss whether what they did would be classified as systematic syncopation or as genuine symbolization.

[94] WALLIS, 1656.

[95] STEDALL, 2004, p. 140.

[96] WALLIS, 1670, p. 394.

[97] Ed. RIGAUD, 1841, pp. 574, 585.

[98] MS Add. 4000, Cambridge University Library, Cambridge, UK, transcription http://www.newtonproject.sussex.ac.uk/view/texts/normalized/NATP00128.

[99] NEWTON, 1707, p. 6 and *passim*.

[100] NEWTON, 1723, pp. 23-25; 1736, p. 18.

[101] The manuscript of Newton's Cambridge lectures on algebra (1673-1683) as polished by his Lucasian successor (ed. WHITESIDE, 1972) uses the same system (at times also a Cartesian brace (thus pp. 82-86, dealing with polynomial division). Already as a young student still under Wallis's strong influence Newton must thus have seen the general parenthesis function to be important though of only occasional use, and he choose his own way, to which he stuck forever after.

In the longer run, however, the road was open after Descartes to such fabulous calculations (to mention but this example) as Euler's development of the infinite fractional infinite product[102]

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\,\&c.,}$$

as the sum

$$\begin{aligned}
&1+z\,(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.)\\
&+z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+5x^{10}+\&c.)\\
&+z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+10x^{11}+\&c.)\\
&+z^4(x^4+x^5+2x^6+3x^7+5x^8+6x^9+9x^{10}+11x^{11}+15x^{12}+\&c.)\\
&+z^5(x^5+x^6+2x^7+3x^8+5x^9+7x^{10}+10x^{11}+13x^{12}+18x^{13}+\&c.)\\
&+z^6(x^6+x^7+2x^8+3x^9+5x^{10}+7x^{11}+11x^{12}+14x^{13}+20x^{14}+\&c.)\\
&+z^7(x^7+x^8+2x^9+3x^{10}+5x^{11}+7x^{12}+11x^{13}+15x^{14}+21x^{15}+\&c.)\\
&+z^8(x^8+x^9+2x^{10}+3x^{11}+5x^{12}+7x^{13}+11x^{14}+15x^{15}+22x^{16}+\&c.)\\
&\qquad\&c.,
\end{aligned}$$

without any intermediate argument – thus expecting the reader to know how to transform $\frac{1}{1-x^nz}$ into an infinite sum, and to be able to grasp how the product of this infinite product of infinite sums could be reduced to an infinite sum of infinite sums. Half a century later (and almost certainly before – I have not searched this period and level systematically) the full use of parentheses could even be presupposed at the much more elementary level represented by the general examination at Saint John's College, Cambridge, whose students were confronted in 1797 with this problem:[103]

$$\frac{123+41\sqrt{x}}{5\sqrt{x}-x}=\frac{20\sqrt{x}+4x}{3-\sqrt{x}}-\frac{2x^2}{(5\sqrt{x}-x)(3-\sqrt{x})}.$$

Why? Why Not?

Why was progress so slow, much more marked by stops than by goes, at times even by regressions? Indeed, why not?

Metaphysical absolute progress is nothing but an illusion, mistaking Ivor Grattan-Guinness's famous polemical "royal road to me" for *the* road. Within the broader practice of ocean trade, colonization and warfare, improved mathematical navigation certainly constituted progress – but

[102] Euler, 1748, I, p. 257.
[103] Ed. Rotherham, 1852, p. 3.

from the point of view of the human chattel brought over the Atlantic or dying on the way, the characterization can be disputed. Even Nunez and Dee, however, had little use for algebra when working on navigational techniques.[104] Until their time, algebra had no social uses outside the environment of those who lived from teaching mathematics. Mathematics, of course, also has its *internal* constraints, and those (like Dardi, Antonio and Benedetto) who understood the subject well would not stoop to the false solutions of irreducible cubics and quartics or Giovanni di Davizzo's advertising of roots as inverse powers. Even they, however, used and developed algebra in view of treating *a particular kind* of problems, and for this kind of problems they had no need to develop neither symbolic operations nor embedding and parenthesis function. Personally (but this is already counterfactual history running wild), they might perhaps have enjoyed it if they had been able to foresee that developing such techniques would have enabled them to discover Euler's theorems about the partition of numbers (or just Descartes geometrical results). However, in the competition for pupils and prestige within the environment of abbacus teaching such things would not have been understood and therefore would not have counted, and in any case it is in the nature of dialectic to react to the situation which is already there – nobody gets the idea of deliberately creating tools for the solution of problems which only *practice of the tools they create for purposes they are aware of* will eventually cause to emerge. Even when Descartes shaped the tools later used by an Euler, he *did not and could not* foresee what they would make possible. He shaped them more or less accidentally within his particular context, and had no reason to use them more than he did, preparing a future he did not know about.

Already Descartes, however, lived in a mathematical future unknown to Stifel and his abbacist predecessors. Like theirs, *his* mathematical world was one where *problems* served as challenges, and where the ability to solve problems was the ground for prestige; but the problems were no longer those of repeated travels with gain, finding a purse and sharing its contents, buying a horse in common, or finding numbers in given ratio fulfilling conditions corresponding to particular algebraic equations. Descartes, Wallis and their kind were not Humanists – Humanism, in its heyday (the fourteenth and fifteenth centuries) had never been interested

[104] Regiomontanus does use some algebra in his *De triangulis* – but he needs nothing beyond simple second-degree Florentine abbacus techniques.

in mathematics. Petrarch, as I observed long ago in a different context,[105] wrote several biographical notices of Archimedes the servant of his king and the great engineer, and he spelled the name more correctly than the university scholars of his time – but in contradiction to these he did not know about any of his works. However, as Humanists discovered after 1500 in the wake of the catastrophic *grand tour d'Italie*[106] of the French artillery and after the discovery of the New World, civic utility if restricted to rhetoric and other *studia humanitatis* was useless, civically and in general; civic utility had to encompass technology and even mathematical competence (as reflected in Hans Holbein's *Ambassadors*). In consequence, the Greek mathematicians became interesting, and the *editiones primae* and the first translations of the Greek mathematicians (beyond Euclid and the *Measurement of the Circle*) were produced. For French Humanists and post-Humanists like Viète, Fermat and Descartes, worthy problems were therefore those inspired by Archimedes, Apollonios and Pappos. Algebra was available to them, known as the art of solving problems. But it needed to be reshaped (and not only because of its Arabic name);[107] and that was what they did.

References

M. Abdeljaouad, 2002, *Le manuscrit mathématique de Jerba: Une pratique des symboles algébriques maghrébins en pleine maturité* (Actes du 7ème Colloque Maghrébin sur l'Histoire des Mathématiques Arabes, Marrakech, 30 mai-1 juin 2002), Marrakech, École Normale Supérieure Marrakech, vol. II, pp. 9-98.

M. Abdeljaouad (ed.), 2004, Ibn al-Hā'im, *Sharh al-Urjūza al-Yasminīya*, Tunis, Publications de l'Association Tunisienne des Sciences Mathématiques.

C. Adam, P. Tannery (eds.), 1897, *Oeuvres* de Descartes, 12 vols., Paris, Léopold Cerf, 1997-1910.

G. Arrighi (ed.), 1967, Antonio de' Mazzinghi, *Trattato di Fioretti* nella trascelta a cura di Mo Benedetto secondo la lezione del Codice L.IV.21 (sec. XV) della Biblioteca degl'Intronati di Siena, Pisa, Domus Galilaeana.

– (ed.), 1987, Paolo Gherardi, *Opera mathematica: Libro di ragioni – Liber habaci*. Codici Magliabechiani Classe XI, nn. 87 e 88 (sec. XIV) della Biblioteca Nazionale di Firenze, Lucca, Pacini-Fazzi.

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[105] Høyrup, 1994, p. 211.

[106] Cf. Biagioli, 1989. As Gibbon points out somewhere, the French mercenaries brought greater havoc to Rome than the barbarians of late Antiquity had ever done.

[107] Its promises as well as the shortcomings of its actual shape are pointed out by Descartes in the *Discours* ed. Adam, Tannery, 1897, VI, pp. 17*f*.

– 2004, *La matematica nell'Età di Mezzo. Scritti scelti*, a cura di F. Barberini, R. Franci, L. Toti Rigatelli, Pisa, ETS.

M. Biagioli, 1989, *The Social Status of Italian Mathematicians, 1450-1600*, «History of Science», XXVII, pp. 41-95.

R. Bombelli, 1572, *L'Algebra*, Bologna, Giovanni Rossi (1572, *impr.* 1579).

B. Boncompagni (ed.), 1857, *Scritti* di Leonardo Pisano matematico del secolo decimoterzo, vol. I: Il *Liber abbaci* di Leonardo Pisano, Roma, Tipografia delle Scienze Matematiche e Fisiche.

– 1862, *Scritti* di Leonardo Pisano matematico del secolo decimoterzo, vol. II: *Practica geometriae* ed *Opusculi*, Roma, Tipografia delle Scienze Matematiche e Fisiche.

E. Bortolotti (ed.), 1929, *L'algebra, opera di Rafael Bombelli da Bologna. Libri IV e V*, Bologna, Zanichelli.

G. Calzoni, G. Cavazzoni (eds.), 1996, Luca Pacioli, *Tractatus mathematicus ad discipulos perusinos*, Città di Castello, Delta Grafica.

G. Cardano, 1539, *Practica arithmetice, et mensurandi singularis*, Milano, Bernardini Calusco.

J. Cassinet, 2001, *Une arithmétique toscane en 1334 en Avignon dans la citè des papes et de leurs banquiers florentins*, in *Commerce et mathématiques du moyen âge à la renaissance, autour de la Méditerranée* (Actes du Colloque International du Centre International d'Histoire des Sciences Occitanes, Beaumont de Lomagne, 13-16 mai 1999), Toulouse, Éditions du C.I.H.S.O., pp. 105-128.

G. Chiarini et al. (eds.), 1972, [Pierpaolo Muscharello], *Algorismus. Trattato di aritmetica pratica e mercantile del secolo XV*, 2 vols., Verona, Banca Commerciale Italiana.

H.T. Colebrooke (ed., trans.), 1817, *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*, London, John Murray.

B. Datta, A.N. Singh, 1962, *History of Hindu Mathematics. A Source Book*, Parts I and II, Bombay, Asia Publishing House.

E. de la Roche, 1520, *Larismethique novellement compose*, Lyon, Constantin Fradin.

M. do Céu Pereira da Silva, 2008, *The Algebraic Contents of Bento Fernandes' "Tratado da arte de arismetica" (1555)*, «Historia Mathematica», XXXV, pp. 190-219.

F. Engels, 1962, *Dialektik der Natur*, in K. Marx, F. Engels, *Werke*, vol. 20, Berlin, Dietz Verlag, pp. 305-570.

L. Euler, 1748, *Introductio in analysin infinitorum*, 2 vols., Lausanne, Bosquet.

R. Franci (ed.), 1983, M° Gilio, *Questioni d'algebra dal Codice L.IX.28 della Biblioteca Comunale di Siena* (Quaderni del Centro Studi della Matematica Medioevale, 6), Siena, Servizio Editoriale dell'Università di Siena.

R. Franci, M. Pancanti (eds.), 1988, Anonimo (sec. XIV), *Il trattato d'algibra dal manoscritto Fond. Prin. II. V. 152 della Biblioteca Nazionale di Firenze* (Quaderni del Centro Studi della Matematica Medioevale, 18), Siena, Servizio Editoriale dell'Università di Siena.

F. Ghaligai, 1521, *Summa de arithmetica*, Firenze.

J. Høyrup, 1993, *On Subtractive Operations, Subtractive Numbers, and Purportedly Negative Numbers in Old Babylonian Mathematics*, «Zeitschrift für Assyriologie und Vorderasiatische Archäologie», LXXXIII, pp. 42-60.

– 1994, *In Measure, Number, and Weight. Studies in Mathematics and Culture*, Albany, State University of New York Press.

– 2002, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*, New York, Springer.
– 2007a, *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*, Basel-Boston-Berlin, Birkhäuser.
– 2007b, *Generosity: No Doubt, but at Times Excessive and Delusive*, «Journal of Indian Philosophy», XXXV, pp. 469-485.
– 2010, *Hesitating Progress – The Slow Development Toward Algebraic Symbolization in Abbacus- and Related Manuscripts, c. 1300 to c. 1550*, in A. Heeffer, M. Van Dyck (eds.), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*, London, College Publications, pp. 3-56.
– forthcoming, *Algebra in Cuneiform. Introduction to an Old Babylonian Geometrical Technique*, Berlin, Edition Open Access (foreseen for 2015). Preprint *Max-Planck-Institut für Wissenschaftsgeschichte. Preprint* 452, Berlin, 2013.
B. Hughes, 1986, *Gerard of Cremona's Translation of al-Khwārizmī's Al-Jabr: A Critical Edition*, «Mediaeval Studies» XLVIII, pp. 211-263.
D. Lamrabet, 1994, *Introduction à l'histoire des mathématiques maghrébines*, Rabat, Édité à compte d'auteur.
A. Marre (ed.), 1880, *Le Triparty en la science des nombres par Maistre Nicolas Chuquet Parisien*, «Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche», XIII, pp. 593-659, 693-814.
B. Moss, 1988, *Chuquet's Mathematical Executor: Could Estienne de la Roche have Changed the History of Algebra?*, in C. Hay (ed.), *Mathematics from Manuscript to Print, 1300-1600*, New York, Oxford University Press, pp. 117-126.
G.H.F. Nesselmann, 1842, *Versuch einer kritischen Geschichte der Algebra. Nach den Quellen bearbeitet*, Erster Theil: *Die Algebra der Griechen*, Berlin, G. Reimer.
I. Newton, 1707, *Arithmetica universalis; sive De compositione et resolutione arithmetica*, London, Benjamin Tooke.
– 1723, *Analysis per quantitatum series, fluxiones, ac differentias*, Amsterdam.
– 1736, *The Method of Fluxions and Infinite Series with its Application to the Geometry of Curve-Lines*. Translated from the author's Latin original not yet made public [...] by John Colson, London, Henry Woodfall/John Nourse.
R. Oughtred, 1648, *Clavis mathematica denuò limata, sive potius fabricate*, London, Thomas Whitaker.
L. Pacioli, 1494, *Summa de Arithmetica Geometria Proportioni et Proportionalita*, Venezia, Paganino de Paganini.
J. Peletier, 1554, *L'algebre*, Lyon, Ian de Tournes.
L. Pieraccini (ed.), 1983, M° Biagio, *Chasi exenplari alla regola dell'algibra nella trascelta a cura di M° Benedetto dal Codice L. VII. 2Q della Biblioteca Comunale di Siena* (Quaderni del Centro Studi della Matematica Medioevale, 5), Siena, Servizio Editoriale dell'Università di Siena.
A. Procissi (ed.), 1954, *I "Ragionamenti d'algebra" di R. Canacci*, «Bollettino dell'Unione Matematica Italiana», III, 9, pp. 300-326, 420-451.
[P. Ramus], 1560, *Algebra*, Paris, Andreas Wechelum.
R. Rashed (ed., trans.), 2013, Abū Kāmil, *Algèbre et analyse Diophantienne*, Berlin-Boston, De Gruyter.
S.J. Rigaud (ed.), 1841, *Correspondence of Scientific Men of the Seventeenth Century,*

Including Letters of Barrow, Flamsteed, Wallis, and Newton, Printed from the Originals, in the Collection of the Right Honourable the Earl of Macclesfield, 2 vols., Oxford, Oxford University Press.

W. Rotherham, 1852, *The Algebraical Algebraic Equation and Problem Papers, Proposed in the Examinations o St. John's College, Cambridge, from the Year 1794 to the Present Time*, Cambridge, Henry Wallis.

C. Rudolff, 1525, *Behend und hübsch Rechnung durch die kunstreichen Regeln Algebra, so gemeincklich die Coss genennt warden*, Straßburg.

A. Simi (ed.), 1994, Anonimo (sec. XIV), *Trattato dell'alcibra amuchabile dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze* (Quaderni del Centro Studi della Matematica Medioevale, 22), Siena, Servizio Editoriale dell'Università di Siena.

D.E. Smith, 1923, *History of Mathematics*, 2 vols., Boston, Ginn, 1923-1925.

J. Stedall (ed., trans.), 2004, John Wallis, *The Arithmetic of Infinitesimals*, New York, Springer.

M. Stifel, 1544, *Arithmetica integra*, Nürnberg, Petreius.

T. Tamborini (ed.), 2011, Girolamo Cardano, *Artis magnae sive de regulis algebraicis liber unus*, Milano, FrancoAngeli.

P. Tannery (ed., trans.), 1893, Diophanti Alexandrini, *Opera omnia* cum graecis commentariis, 2 vols., Leipzig, Teubner, 1893-1895.

N. Tartaglia, 1556, *La seconda parte del general trattato di numeri, et misure*, Venezia, Curtio Troiano.

– 1560, *La sesta parte del general trattato de' numeri, et misure*, Venezia, Curtio Troiano.

E. Ulivi, 1996, *Per una biografia di Antonio Mazzinghi, maestro d'abaco del XIV secolo*, «Bollettino di Storia delle Scienze Matematiche», XVI, pp. 101-150.

W. van Egmond, 1983, *The Algebra of Master Dardi of Pisa*, «Historia Mathematica», X, pp. 399-421.

C. van Randenborgh, 2012, *Franz van Schootens Beitrag zu Descartes Discours de la method*, «Mathematische Semesterberichte», LXIX, pp. 233-241.

F. van Schooten (ed.), 1646, François Viète, *Opera mathematica*, Leiden, Elzevier.

A. Vasset, 1630, *L'algebre nouvelle de M^r. Viète*, Paris, Pierre Rocolet.

J.-L. Vaulezard, 1630, *Les cinq livres des zetetiques de Francois Viette*, Paris, Julian Jacquin.

F. Viète, 1591, *In artem analyticem isagoge*, Tours, J. Mettayer.

– 1593, *Zeteticorum libri V*, Tours, J. Mettayer.

A.-M. Vlasschaert (ed.), 2010, *Liber Mahamaleth*, Wiesbaden, Franz Steiner.

K. Vogel, 1930, *Die algebraischen Probleme des P. Mich. 620*, «Classical Philology», XXV, pp. 373-375.

J. Wallis, 1656, *Arithmetica infinitorum*, Oxford, Thomas Robinson.

– 1670, *Mechanica: sive, De Motu, Tractatus Geometricus*, London, Moses Pitt.

A. Weil, 1978, *Who Betrayed Euclid?*, «Archive for History of Exact Sciences», XIX, pp. 91-93.

P. Wendland (ed.), 1916, Hippolytus *Werke*. Vol. III. *Refutatio omnium haeresium*, Leipzig, Hinrichs'sche Buchhandlung.

D.T. Whiteside (ed., trans.), 1972, *The Mathematical Papers* of Isaac Newton, vol. V, 1683-1684, Cambridge, Cambridge University Press.

T.R. WITMER (ed., trans.), 1983, François Viète, *The Analytic Art: Nine Studies in Algebra, Geometry, and Trigonometry from the Opus restitutae mathematicae analyseos, seu algebrâ nova*, Kent, OH, The Kent State University Press.

F. WOEPCKE, 1853, *Extrait du Fakhrî, traité d'algèbre* par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; *précédé d'un mémoire sur l'algèbre indéterminé chez les Arabes*, Paris, L'Imprimerie Impériale.

– 1854, *Recherches sur l'histoire des sciences mathématiques chez les Orientaux, d'après des traités inédits arabes et persans. Premier article. Notice sur des notations algébriques employées par les Arabes*, «Journal Asiatique», 5e série, IV, pp. 348-384.

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